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LETTER TO THE EDITOR

Similarity reduction of a (2 + 1) Volterra system

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Abstract. The continuous point symmetries of a (2 + 1) discrete Volterra system is derived and it forms a centreless infinite-dimensional Kac–Moody Virasoro algebra. Using the symmetries, a similarity reduction of the (2 + 1) Volterra system to a partial differential-difference equation with two independent variables is obtained. Also, the Lie point symmetries of the reduced partial differential-difference equation is derived and its similarity reduction to an ordinary differential-difference equation is obtained. The ordinary differential-difference equation reduces to the third Painlevé equation in the continuum limit. Further, a Lax pair for the reduced equation is derived indicating its complete integrability.

It is well known that the discovery of Norwegian mathematician Sophus Lie at the beginning of the nineteenth century on the integration theory of differential equations has played a vital role in investigations into different mathematical aspects of soliton systems governed by continuous equations during the past few decades. The primary objective of the Lie symmetry analysis advocated by Sophus Lie is to find one- or several-parameter local continuous transformations leaving the equations invariant and then exploit them to obtain the so-called invariant or similarity solutions, invariants, integrals of motion, etc [1–3], and the usefulness of this approach has been widely illustrated by several authors in different contexts [4–8]. This method, essentially to derive Lie symmetries of continuous systems (differential equations), has been recently extended to discrete systems governed by differential-difference and difference equations and it has been shown how to derive the continuous symmetries and the associated group-theoretical properties [9–16].

It is of interest in discrete nonlinear systems to analyse the nature of equations (or solutions) in higher dimensions, whose lower-dimensional counterparts are solvable by inverse scattering transform methods. Ablowitz *et al* [17] have conjectured that every ordinary differential equation obtained by similarity reduction of an inverse scattering solvable partial differential equation is completely integrable. In [11] one of the authors of this letter has shown that the above conjecture holds for (1 + 1) inverse scattering transform solvable partial differential-difference equations as well. In this letter, we investigate whether such a result holds for higher-dimensional discrete nonlinear systems. In particular, we wish to show that the similarity reduction of a (2 + 1) Volterra system

$$\frac{\partial}{\partial t}c_n + \sigma^2 \frac{\partial}{\partial y} w_n = c_n(c_{n-1}^2 - c_{n+1}^2) \tag{1a}$$

$$\frac{\partial}{\partial y}(c_n c_{n-1}) = c_n w_{n-1} - c_{n-1} w_n \tag{1b}$$

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where $\sigma^2 = \pm 1$, $c_n = c(n, y, t)$, $w_n = w(n, y, t)$, y and t are continuous variables and n is a discrete variable, admits a Lax pair indicating its complete integrability. We also show that the similarity reduction reduces to the third Painlevé equation in the continuum limit. We wish to mention that the $(2+1)$ Volterra system (1a), (1b) is solvable by the inverse scattering transform [18] and that the continuum limit of it reduces to the well known Kadomtsev–Petviashvile equation while its $(1+1)$ case ($w_n = 0$, $c_n = c(n, t)$) reduces to the well known Volterra system

$$\frac{\partial}{\partial t} c_n = c_n(c_{n-1}^2 - c_{n+1}^2) \tag{2}$$

and is solvable by the inverse scattering transform technique.

Consider a one-parameter (ϵ) Lie group of continuous point transformations,

$$n^* = n \qquad t^* = t + \epsilon\xi_1 + \mathrm{O}(\epsilon^2) \qquad y^* = y + \epsilon\xi_2 + \mathrm{O}(\epsilon^2) \tag{3a}$$

$$c_{n^*}^* = c_n + \epsilon\xi_3 + \mathrm{O}(\epsilon^2) \qquad w_{n^*}^* = w_n + \epsilon\xi_4 + \mathrm{O}(\epsilon^2) \tag{3b}$$

where $\xi_i = \xi_i(n, t, y, c_n, w_n)$, $i = 1, 2, 3, 4$, are infinitesimals and the associated infinitesimal generator is

$$X = \xi_1(n, y, t)\frac{\partial}{\partial t} + \xi_2(n, y, t)\frac{\partial}{\partial y} + \xi_3(n, y, t)\frac{\partial}{\partial c_n} + \xi_4(n, y, t)\frac{\partial}{\partial w_n}. \tag{3c}$$

The $(2+1)$ Volterra system, equation (1), is invariant under the transformations (3) if

$$\frac{\partial}{\partial t^*} c_{n^*}^* + \sigma^2 \frac{\partial}{\partial y^*} w_{n^*}^* = c_{n^*}^*(c_{n^*-1}^{*2} - c_{n^*+1}^{*2}) \tag{4a}$$

$$\frac{\partial}{\partial y^*}(c_{n^*}^* c_{n^*-1}^*) = c_{n^*}^* w_{n^*-1}^* - c_{n^*-1}^* w_{n^*}^* \tag{4b}$$

provided c_n and w_n satisfy equations (1a), (1b). Making use of the expressions for $\frac{\partial}{\partial t^*} c_{n^*}^*$, $\frac{\partial}{\partial y^*} c_{n^*}^*$ and $\frac{\partial}{\partial y^*} w_{n^*}^*$ [2, 3] in equation (4), after some calculations we find $\xi_1(n, y, t, c_n, w_n) = \xi_1(t)$, $\xi_2(n, y, t, c_n, w_n) = \xi_2(y, t)$, $\xi_3(n, y, t, c_n, w_n) = \xi_3(t, c_n)$, $\xi_4(n, y, t, c_n, w_n) = \xi_4(y, t, c_n, w_n)$ and the following invariance equation:

$$\begin{aligned}(\xi_{3,t} + \sigma^2\xi_{4,y}) &- (\xi_{2,t} - \sigma^2\xi_{4,c_n})c_{n,y} + (\xi_{3,c_n} - \xi_{1,t})c_{n,t} + \sigma^2(\xi_{4,w_n} - \xi_{2,y})w_{n,y} \\ &-(c_{n-1}^2 - c_{n+1}^2)\xi_3 - 2c_n(c_{n-1}\xi_3(n-1) - c_{n+1}\xi_3(n+1)) = 0\end{aligned} \tag{5a}$$

and

$$\begin{aligned}\xi_3 c_{n-1,y} &+ \xi_3(n-1)c_{n,y} + (\xi_{3,c_{n-1}}(n-1) - \xi_{2,y}(n-1))c_n c_{n-1,y} \\ &+(\xi_{3,c_n} - \xi_{2,y})c_{n-1}c_{n,y} - c_n\xi_4(n-1) - \xi_3 w_{n-1} + \xi_4 c_{n-1} \\ &+\xi_3(n-1)w_n = 0\end{aligned} \tag{5b}$$

where subscripts denote partial differentation, that is $\xi_{3,t} = \frac{\partial\xi_3}{\partial t}$, $c_{n,t} = \frac{\partial c_n}{\partial t}$, $\xi_{3,y}(n-1) = \frac{\partial\xi_3(n-1)}{\partial y}$, etc.

Solving equations (5a), (5b) we obtain

$$\xi_1(n, y, t) = h(t) \qquad \xi_2(n, y, t) = \tfrac{1}{2}\dot{h}(t)y + \sigma^2 k(t) \tag{6a}$$

$$\xi_3(n, y, t) = -\tfrac{1}{2}\dot{h}(t)c_n \qquad \xi_4(n, y, t) = [\dot{k}(t) + \tfrac{1}{2}\sigma^2 y\ddot{h}(t)]c_n - \dot{h}(t)w_n \tag{6b}$$

where $h(t)$, $k(t)$ are arbitrary functions and $\dot{h} = \frac{\mathrm{d}h}{\mathrm{d}t}$ and so the infinitesimal generator (3c) takes the following form:

$$\begin{aligned}X = h(t)\frac{\partial}{\partial t} &+ \frac{1}{2}y\dot{h}(t)\frac{\partial}{\partial y} - \frac{1}{2}\dot{h}(t)c_n\frac{\partial}{\partial c_n} + \left[\frac{1}{2}\sigma^2 y\ddot{h}(t)c_n - \dot{h}(t)w_n\right]\frac{\partial}{\partial w_n} \\ &+\sigma^2 k(t)\frac{\partial}{\partial y} + \dot{k}(t)c_n\frac{\partial}{\partial_{wn}}.\end{aligned} \tag{7}$$

It is straightforward to check that the commutators $X_1(h)$ and $X_2(k)$ satisfy the following relations:

$$[X_1(h_1), X_1(h_2)] = X_1(h_1\dot{h}_2 - h_2\dot{h}_1)$$
$$[X_1(h), X_2(k)] = X_2(h\dot{k} - \tfrac{1}{2}k\dot{h})$$
$$[X_2(k_1), X_2(k_2)] = 0$$

and thus the underlying symmetry algebra is of centreless Kac–Moody–Virasoro type. Thus the $(2+1)$ Volterra system, equations (1*a*), (1*b*) with two continuous independent variables (y, t) and one discrete independent variable n shares a property so far observed only for integrable nonlinear systems with three continuous independent variables. A similar observation for the $(2+1)$ Toda lattice system was pointed out by Levi and Winternitz [10].

Next, the similarity variable and the similarity transformation associated with the above set of symmetries $(\xi_1, \xi_2, \xi_3, \xi_4)$, equations (6*a*) and (6*b*), can be obtained by solving the characteristic equation

$$\frac{\mathrm{d}t}{\xi_1} = \frac{\mathrm{d}y}{\xi_2} = \frac{\mathrm{d}c_n}{\xi_3} = \frac{\mathrm{d}w_n}{\xi_4}. \tag{8}$$

For clarity of presentation we consider $h(t)$ to be linear in t and $k(t) = 0$ (the derivation of similarity variables and similarity transformation for arbitrary $h(t)$ and $k(t)$ is a straightforward one) and so the infinitesimal symmetries (6*a*), (6*b*) become

$$\xi_1 = at + a_1 \qquad \xi_2 = \tfrac{1}{2}ay \qquad \xi_3 = -\tfrac{1}{2}ac_n \qquad \xi_4 = -aw_n \tag{9}$$

where a, a_1 are arbitrary constants.

Solving the above characteristic equation (8) we obtain the following similarity variables:

$$\eta_1 = n \qquad \text{and} \qquad \eta_2 = \frac{y}{(at + a_1)^{\frac{1}{2}}} \tag{10a}$$

and the similarity transformations $f(\eta_1, \eta_2)$ and $g(\eta_1, \eta_2)$ take

$$f(\eta_1, \eta_2) = (at + a_1)^{\frac{1}{2}} c_n \qquad \text{and} \qquad g(\eta_1, \eta_2) = (at + a_1)w_n. \tag{10b}$$

Substituting the similarity transformations, equation (10*b*) along with similarity variables (10*a*) in the Volterra system, equations (1*a*) and (1*b*), we find that it reduces into a partial differential-difference equation with two independent variables (η_1, η_2) (one discrete, one continuous):

$$a\eta_2 \frac{\partial f}{\partial \eta_2} + af - 2\sigma^2 \frac{\partial g}{\partial \eta_2} = 2f(\eta_1, \eta_2)[f^2(\eta_1 + 1, \eta_2) - f^2(\eta_1 - 1, \eta_2)] \tag{11a}$$

and

$$\frac{\partial}{\partial \eta_2}[f(\eta_1, \eta_2)f(\eta_1 - 1, \eta_2)] = f(\eta_1, \eta_2)g(\eta_1 - 1, \eta_2) - g(\eta_1, \eta_2)f(\eta_1 - 1, \eta_2). \tag{11b}$$

Now again treating equations (11*a*), (11*b*) as a partial differential-difference equation in two independent variables η_1 and η_2, we make another infinitesimal transformation in η_1, η_2, f and g

$$\eta_1^* = \eta_1 + \epsilon\phi_1 + \mathrm{O}(\epsilon^2) \qquad \eta_2^* = \eta_2 + \epsilon\phi_2 + \mathrm{O}(\epsilon^2) \tag{12a}$$
$$f^*(\eta_1^*, \eta_2^*) = f(\eta_1, \eta_2) + \epsilon\phi_3 + \mathrm{O}(\epsilon^2) \qquad g^*(\eta_1^*, \eta_2^*) = g(\eta_1, \eta_2) + \epsilon\phi_4 + \mathrm{O}(\epsilon^2) \tag{12b}$$

where $\phi_i = \phi_i(\eta_1, \eta_2, f, g)$, $i = 1, 2, 3, 4$ are infinitesimals and the infinitesimal generator Y is

$$Y = \phi_1 \frac{\partial}{\partial \eta_1} + \phi_2 \frac{\partial}{\partial \eta_2} + \phi_3 \frac{\partial}{\partial f} + \phi_4 \frac{\partial}{\partial g}. \tag{12c}$$

Equations (11a), (11b) are invariant under the transformations (12a), (12b) if

$$a\eta_2^*\frac{\partial f^*}{\partial\eta_{2^*}} + af^* - 2\sigma^2\frac{\partial g^*}{\partial\eta_{2^*}} = 2f^*(\eta_1^*,\eta_2^*)[f^{*^2}(\eta_1^*+1,\eta_2^*) - f^{*^2}(\eta_1^*-1,\eta_2^*)] \tag{13a}$$

and

$$\frac{\partial}{\partial\eta_2^*}[f^*(\eta_1^*,\eta_2^*)f^*(\eta_1^*-1,\eta_2^*)] = f^*(n_1^*,n_2^*)g^*(n_1^*-1,n_2^*) - g^*(\eta_1^*,\eta_2^*)f^*(\eta_1^*-1,\eta_2^*) \tag{13b}$$

provided $f(\eta_1,\eta_2)$ and $g(\eta_1,\eta_2)$ satisfy equations (11).

Proceeding as before we find that for the following infinitesimal symmetries:

$$\phi_1 = -\beta \qquad \phi_2 = b\eta_2 + b_1 \qquad \phi_3 = -bf \qquad \phi_4 = -2bg + \frac{a}{\sigma^2}\left(b\eta_2 + \frac{b_1}{2}\right)f \tag{14}$$

equations (13a,b) satisfy simultaneously where β, b, b_1 are arbitrary constants and the infinitesimal generator Y, equation (12c), becomes

$$Y = -\beta\frac{\partial}{\partial\eta_1} + (b\eta_2+b_1)\frac{\partial}{\partial\eta_2} - bf\frac{\partial}{\partial f} + \left[-2bg + \frac{a}{\sigma^2}\left(b\eta_2+\frac{b_1}{2}\right)f\right]\frac{\partial}{\partial g}. \tag{15}$$

Here the generators Y_1, Y_2 and Y_3 become

$$Y_1 = -\frac{\partial}{\partial\eta_1} \qquad Y_2 = \frac{\partial}{\partial\eta_2} + \frac{a}{2\sigma^2}f\frac{\partial}{\partial g}$$

$$Y_3 = \eta_2\frac{\partial}{\partial\eta_2} - f\frac{\partial}{\partial f} + \left[-2g + \frac{a}{\sigma^2}f\eta_2\right]\frac{\partial}{\partial g}$$

and the commutation relation satisfies

$$[Y_1, Y_2] = 0 \qquad [Y_1, Y_3] = 0 \qquad [Y_2, Y_3] = Y_2$$

indicating that the underlying symmetry algebra of (11a), (11b) is nilpotent.

In order to obtain the similarity variable and the similarity transformation associated with the above set of symmetries ($\phi_1, \phi_2, \phi_3, \phi_4$), equation (14), we solve the Lagrange characteristic equation

$$\frac{\mathrm{d}\eta_1}{-\beta} = \frac{\mathrm{d}\eta_2}{b\eta_2+b_1} = \frac{\mathrm{d}f}{-bf} = \frac{\mathrm{d}g}{-2bg + \frac{a}{\sigma^2}(b\eta_2+\frac{b_1}{2})f}. \tag{16}$$

Solving the above characteristic equation (16) we obtain a similarity variable,

$$\zeta = \eta_1 + \frac{\beta}{b}\log(b\eta_2+b_1) \tag{17a}$$

and the similarity transformations $F(\zeta)$ and $G(\zeta)$ become

$$F(\zeta) = (b\eta_2+b_1)f \qquad G(\zeta) = \left[g - \frac{a}{2\sigma^2}\eta_2 f\right](b\eta_2+b_1)^2. \tag{17b}$$

Substituting the similarity transformations, equation (17b), along with the similarity variable, equation (17a), we find that the partial differential-difference equation, equations (11a) and (11b), reduces into an ordinary differential-difference equation:

$$\sigma^2\left[\beta\frac{\mathrm{d}G}{\mathrm{d}\zeta} - 2bG(\zeta)\right] = F(\zeta)[F^2(\zeta-1) - F^2(\zeta+1)] \tag{18a}$$

and

$$\beta\frac{\mathrm{d}}{\mathrm{d}\zeta}[F(\zeta)F(\zeta-1)] - 2bF(\zeta)F(\zeta-1) = F(\zeta)G(\zeta-1) - G(\zeta)F(\zeta-1). \tag{18b}$$

Next, it is known that the (2 + 1) Volterra system (1*a*), (1*b*) possesses Lax representation and so is solvable through the inverse scattering transform technique [18]. For $\sigma^2 = 1$ the Lax representation for the similarity reduction of the (2 + 1) Volterra system becomes

$$\left(\beta\frac{\mathrm{d}}{\mathrm{d}\zeta} + 2b\lambda\frac{\mathrm{d}}{\mathrm{d}\lambda}\right) R(\zeta,\lambda) = F(\zeta)R(\zeta-1,\lambda) - F(\zeta+1)R(\zeta+1,\lambda) \tag{19a}$$

$$\begin{aligned}\lambda R(\zeta,\lambda) = {} & G(\zeta)R(\zeta-1,\lambda) - G(\zeta+1)R(\zeta+1,\lambda) + F(\zeta)F(\zeta-1)R(\zeta-2,\lambda)\\ & -F(\zeta+1)F(\zeta+2)R(\zeta+2,\lambda)\end{aligned} \tag{19b}$$

where λ is a spectral parameter. The compatibility condition of equations (19*a*), (19*b*) gives the reduced equations (18*a*), (18*b*). Thus the reduced equations (19*a*), (19*b*) are expected to be integrable. A similar conclusion can be drawn for the case $\sigma^2 = -1$.

To find the continuum limit of the reduced equations (18*a*), (18*b*) we introduce the following transformations:

$$F(\zeta) = \mathrm{e}^{\frac{\phi(\zeta)}{2}} \qquad G(\zeta) = \mathrm{e}^{\frac{\phi(\zeta)}{2}} A(\zeta)$$

and so equations (18*a*), (18*b*) become

$$\sigma^2\left[\beta\left(\frac{1}{2}A(\zeta)\frac{\mathrm{d}}{\mathrm{d}\zeta}\phi(\zeta) + \frac{\mathrm{d}}{\mathrm{d}\zeta}A(\zeta)\right) - 2bA(\zeta)\right] = \mathrm{e}^{\phi(\zeta-1)} - \mathrm{e}^{\phi(\zeta+1)} \tag{20a}$$

$$\frac{\beta}{2}\left[\frac{\mathrm{d}}{\mathrm{d}\zeta}(\phi(\zeta)+\phi(\zeta-1))\right] - 2b = A(\zeta-1) - A(\zeta). \tag{20b}$$

The above equations (20*a*), (20*b*) can be rewritten as

$$\begin{aligned}&\frac{\beta\sigma 2}{2}\left[\beta\frac{\mathrm{d}^2}{\mathrm{d}\zeta^2}(\phi(\zeta)+\phi(\zeta-1)) + A(\zeta-1)\frac{\mathrm{d}}{\mathrm{d}\zeta}\phi(\zeta-1) - A(\zeta)\frac{\mathrm{d}}{\mathrm{d}\zeta}\phi(\zeta)\right]\\ &\qquad +2b\sigma^2[A(\zeta) - A(\zeta-1)] = \mathrm{e}^{\phi(\zeta+1)} - \mathrm{e}^{\phi(\zeta-1)} + \mathrm{e}^{\phi(\zeta-2)} - \mathrm{e}^{\phi(\zeta)}.\end{aligned} \tag{21}$$

By choosing $\phi(\zeta) = 1 + \epsilon v(\zeta)$, $A(\zeta) = \frac{-2}{\sigma^2}[4 - \sigma^2 + v(\zeta)\zeta]$, $\beta = \epsilon$ and $b = \epsilon^3$ and then substituting in equation (21) we find that it reduces to

$$\frac{\mathrm{d}^2 v}{\mathrm{d}\zeta^2} + \frac{1}{\zeta}\frac{\mathrm{d}v}{\mathrm{d}\zeta} + \frac{1}{v}\left(\frac{\mathrm{d}v}{\mathrm{d}\zeta}\right)^2 = 0$$

which is the third Painlevé equation.

The derivation of conditional symmetries and generalized symmetries including Lie Backlund symmetries of the Volterra system (1*a*), (1*b*) as well as the details of the singularity confinement criterion of (18*a*), (18*b*) (a discrete version of the Painlevé property) proposed by Grammaticos *et al* [19] will be published elsewhere.

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